

Phases and phase transitions in spin-triplet ferromagnetic superconductors

D. V. Shopova and D. I. Uzunov^{*,†}

*CPCM Laboratory, G. Nadjakov Institute of Solid State Physics,
Bulgarian Academy of Sciences, BG-1784 Sofia, Bulgaria.*

* Also, Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany.

† Corresponding author: uzun@issp.bas.bg

Key words: superconductivity, ferromagnetism, phase diagram, order parameter profile.

PACS: 74.20.De, 74.20.Rp

Abstract

Recent results for the coexistence of ferromagnetism and unconventional superconductivity with spin-triplet Cooper pairing are reviewed on the basis of the quasi-phenomenological Ginzburg-Landau theory. New results are presented for the properties of phases and phase transitions in such ferromagnetic superconductors. The superconductivity, in particular, the mixed phase of coexistence of ferromagnetism and unconventional superconductivity is triggered by the spontaneous magnetization. The mixed phase is stable whereas the other superconducting phases that usually exist in unconventional superconductors are either unstable, or, for particular values of the parameters of the theory, some of these phases are metastable at relatively low temperatures in a quite narrow domain of the phase diagram. The phase transitions from the normal phase to the phase of coexistence is of first order while the phase transition from the ferromagnetic phase to the coexistence phase can be either of first or second order depending on the concrete substance. The Cooper pair and crystal anisotropies are relevant to a more precise outline of the phase diagram shape and reduce the degeneration of the ground states of the system but they do not drastically influence the phase stability domains and the thermodynamic properties of the respective phases. The results are discussed in view of application to metallic ferromagnets as UGe_2 , ZrZn_2 , URhGe .

1. Introduction

1.1. Notes about unconventional superconductivity

The phenomenon of unconventional Cooper pairing of fermions, i.e. the formation of Cooper pairs with nonzero angular momentum was theoretically predicted [1] in

1959 as a mechanism of superfluidity of Fermi liquids. In 1972 the same phenomenon - unconventional superfluidity due to a p -wave (spin triplet) Cooper pairing of ^3He atoms, was experimentally discovered in the mK range of temperatures; for details and theoretical description, see Refs. [2, 3, 4]. Note that, in contrast to the standard s -wave pairing in usual (conventional) superconductors, where the electron pairs are formed by an attractive electron-electron interaction due to a virtual phonon exchange, the widely accepted mechanism of the Cooper pairing in superfluid ^3He is based on an attractive interaction between the fermions (^3He atoms) due to a virtual exchange of spin fluctuations. Certain spin fluctuation mechanisms of unconventional Cooper pairing of electrons have been assumed also for the discovered in 1979 heavy fermion superconductors (see, e.g., Refs. [5, 6, 7]) as well as for some classes of high-temperature superconductors (see, e.g., Refs. [8, 9, 10, 11, 12, 13, 14, 15, 16]).

The possible superconducting phases in unconventional superconductors are described in the framework of the general Ginzburg-Landau (GL) effective free energy functional [13] with the help of the symmetry groups theory. Thus a variety of possible superconducting orderings were predicted for different crystal structures [17, 18, 19, 20, 21, 22]. A detailed thermodynamic analysis [11, 18] of the homogeneous (Meissner) phases and a renormalization group investigation [11] of the superconducting phase transition up to the two-loop approximation have been also performed (for a three-loop renormalization group analysis, see Ref. [23]; for effects of magnetic fluctuations and disorder, see [24, 25]). We shall essentially use these results in our present consideration.

In 2000, experiments [26] at low temperatures ($T \sim 1$ K) and high pressure ($T \sim 1$ GPa) demonstrated the existence of spin triplet superconducting states in the metallic compound UGe_2 . This superconductivity is triggered by the spontaneous magnetization of the ferromagnetic phase which exists at much higher temperatures and coexists with the superconducting phase in the whole domain of existence of the latter below $T \sim 1$ K; see also experiments published in Refs. [27, 28], and the discussion in Ref. [29]. Moreover, the same phenomenon of existence of superconductivity at low temperatures and high pressure in the domain of the (T, P) phase diagram where the ferromagnetic order is present has been observed in other ferromagnetic metallic compounds (ZrZn_2 [30] and URhGe [31]) soon after the discovery [26] of superconductivity in UGe_2 .

In contrast to other superconducting materials, for example, ternary and Chevrel phase compounds, where the effects of magnetic order on superconductivity are also substantial (see, e.g., [32, 33, 34, 35]), in these ferromagnetic compounds the phase transition temperature (T_f) to the ferromagnetic state is much higher than the phase transition temperature (T_{FS}) from ferromagnetic to a (mixed) state of coexistence of ferromagnetism and superconductivity. For example, in UGe_2 we have $T_{FS} = 0.8$ K whereas the critical temperature of the phase transition from paramagnetic to ferromagnetic state in the same material is $T_f = 35$ K [26, 27]. One may reliably assume that in such kind of

materials the material parameter T_s defined as the (usual) critical temperature of the second order phase transition from normal to uniform (Meissner) superconducting state in zero external magnetic field is quite lower than the phase transition temperature T_{FS} . Note, that the mentioned experiments on the compounds UGe_2 , URhGe , and ZrZn_2 do not give any evidence for the existence of a standard normal-to-superconducting phase transition in zero external magnetic field.

Moreover, it seems that the superconductivity in the metallic compounds mentioned above, always coexists with the ferromagnetic order and is enhanced by the latter. As claimed in Ref. [26] in these systems the superconductivity seems to arise from the same electrons that create the band magnetism, and is most naturally understood as a triplet rather than spin-singlet pairing phenomenon. Note that all three metallic compounds, mentioned so far, are itinerant ferromagnets. Besides, the unconventional superconductivity has been suggested [36] as a possible outcome of recent experiments in Fe [37], in which a superconducting phase was discovered at temperatures below 2 K at pressures between 15 and 30 GPa. Note, that both vortex and Meissner superconductivity phases [37] are found in the high-pressure crystal modification of Fe which has a hexagonal close-packed lattice. In this hexagonal lattice the strong ferromagnetism of the usual bcc iron crystal probably disappears [36]. Thus one can hardly claim that there is a coexistence of ferromagnetism and superconductivity in Fe but the clear evidence for a superconductivity is also a remarkable achievement.

1.2. Ferromagnetism versus superconductivity

The important point in all discussions of the interplay of superconductivity and ferromagnetism is that a small amount of magnetic impurities can destroy superconductivity in conventional (s -wave) superconductors by breaking up the (s -wave) electron pairs with opposite spins (paramagnetic impurity effect [38]). In this aspect the phenomenological arguments [39] and the conclusions on the basis of the microscopic theory of magnetic impurities in s -wave superconductors [38] are in a complete agreement with each other; see, e.g., Refs. [32, 33, 34, 35]. In fact, a total suppression of conventional (s -wave) superconductivity should occur in the presence of an uniform spontaneous magnetization \mathbf{M} , i.e. in a standard ferromagnetic phase [39]. The physical reason for this suppression is the same as in the case of magnetic impurities, namely, the opposite electron spins in the s -wave Cooper pair turn over along the vector \mathbf{M} in order to lower their Zeeman energy and, hence, the pairs break down. Therefore, the ferromagnetic order can hardly coexist with conventional superconducting states. In particular, this is the case of coexistence of uniform superconducting and ferromagnetic states when the superconducting order parameter $\psi(\mathbf{x})$ and the magnetization \mathbf{M} do not depend on the spatial vector \mathbf{x} .

But yet a coexistence of s -wave superconductivity and ferromagnetism may appear in uncommon materials and under quite special circumstances. Furthermore, let us emphasize that the conditions for the coexistence of nonuniform (“vertex”, “spiral”, “spin-sinoidal” or “helical”) superconducting and ferromagnetic states are less re-

strictive than that for the coexistence of uniform superconducting and ferromagnetic orders. Coexistence of nonuniform phases has been discussed in details, in particular, experiment and theory of ternary and Chevrel-phase compounds, where such a coexistence seems quite likely; for a comprehensive review, see, for example, Refs. [32, 33, 34, 35, 40].

In fact, the only two superconducting systems for which the experimental data allow assumptions in a favor of a coexistence of superconductivity and ferromagnetism are the rare earth ternary boride compound ErRh_4B_4 and the Chevrel phase compound HoMo_6S_8 ; for a more extended review, see Refs. [33, 41]. In these compounds the phase of coexistence most likely appears in a very narrow temperature region just below the Curie temperature T_f of the ferromagnetic phase transition. At lower temperatures the magnetic moments of the rare earth $4f$ electrons become better aligned, the magnetization increases and the s -wave superconductivity pairs formed by the conduction electrons disintegrate.

1.3. Unconventional superconductivity triggered by ferromagnetic order

We shall not extend our consideration over all important aspects of the long standing problem of coexistence of superconductivity and ferromagnetism rather we shall concentrate our attention on the description of the newly discovered coexistence of ferromagnetism and unconventional (spin-triplet) superconductivity in the itinerant ferromagnets UGe_2 , ZrZn_2 , and URhGe . Here we wish to emphasize that the main object of our discussion is the superconductivity of these compounds and, at a second place in the rate of importance we put the problem of coexistence. The reason is that the existence of superconductivity in such itinerant ferromagnets is a highly nontrivial phenomenon. As noted in Ref. [42] the superconductivity in these materials seems difficult to explain in terms of previous theories [32, 33, 35] and seems to require new concepts to interpret the experimental data.

We have already mentioned that in ternary compounds the ferromagnetism comes from the localized $4f$ electrons whereas the s -wave Cooper pairs are formed by conduction electrons. In UGe_2 and URhGe the $5f$ electrons of U atoms form both superconductivity and ferromagnetic order [26, 31]. In ZrZn_2 the same double role is played by the $4d$ electrons of Zr. Therefore the task is to describe this behavior of the band electrons at a microscopic level. One may speculate about a spin-fluctuation mediated unconventional Cooper pairing as is in case of ^3He and heavy fermion superconductors. These important issues have not yet a reliable answer and for this reason we shall confine our consideration to a phenomenological level.

In fact, a number of reliable experimental data for example, the data about the coherence length and the superconducting gap [26, 27, 31, 30], are in favor of the conclusion about a spin-triplet Cooper pairing in these metallic compounds, although the mechanism of this pairing remains unclear. We shall essentially use this reliable conclusion. Besides, this point of view is consistent with the experimental observation of coexistence

of superconductivity only in a low temperature part of the ferromagnetic domain of the phase diagram (T, P) , which means that a pure (non ferromagnetic) superconducting phase has not been observed. This circumstance is also in favor of the assumption of a spin-triplet superconductivity. Our investigation leads to results which confirm this general picture.

Besides, on the basis of the experimental data and conclusions presented for the first time in Refs. [26, 29] and shortly afterwards confirmed in Refs. [27, 28, 30, 31] one may reliably accept the point of view that the the superconductivity in these magnetic compounds is considerably enhanced by the ferromagnetic order parameter \mathbf{M} and, perhaps, it could not exist without this “mechanism of ferromagnetic trigger,” or, in short, “ \mathbf{M} -trigger.” Such a phenomenon is possible for spin-triplet Cooper pairs, where the electron spins point parallel to each other and their turn along the vector of the spontaneous magnetization \mathbf{M} does not produce a break down of the spin-triplet Cooper pairs but rather stabilizes them and, perhaps, stimulates their creation. We shall describe this phenomenon at a phenomenological level.

1.4. Phenomenological studies

Recently, the phenomenological theory which explains the coexistence of ferromagnetism and unconventional spin-triplet superconductivity of Landau-Ginzburg type was developed [42, 43]. The possible low-order couplings between the superconducting and ferromagnetic order parameters were derived with the help of general symmetry group arguments and several important features of the superconducting vortex state in the ferromagnetic phase of unconventional ferromagnetic superconductors were established [42, 43].

In this article we shall use the approach presented in Refs. [42, 43] to investigate the conditions for the occurrence of the Meissner phase and to demonstrate that the presence of ferromagnetic order enhances the p -wave superconductivity. Besides, we shall establish the phase diagram corresponding to model ferromagnetic superconductors in a zero external magnetic field. We shall show that the phase transition to the superconducting state in ferromagnetic superconductors can be either of first or second order depending on the particular substance. We confirm the predictions made in Refs. [42, 43] about the symmetry of the ordered phases.

Our investigation is based on the mean-field approximation [13] as well as on familiar results about the possible phases in nonmagnetic superconductors with triplet (p -wave) Cooper pairs [18, 11, 12]. Results from Refs. [44, 45, 46] will be reviewed and extended. In our preceding investigation [44, 45, 46] both Cooper pair anisotropy and crystal anisotropy have been neglected in order to clarify the main effect of the coupling between the ferromagnetic and superconducting order parameters. The phenomenological GL free energy is quite complex and the inclusion of these anisotropies is related with lengthy formulae and a multivariant analysis which obscures the final results.

Here we shall take into account essential anisotropy effects, in particular, the effect of

the Cooper pair anisotropy on the existence and stability of the mixed phase, namely the phase of coexistence of superconductivity and ferromagnetic order. We demonstrate that the anisotropy of the spin-triplet Cooper pairs modifies but does not drastically change the thermodynamic properties of this coexistence phase, in particular, in the most relevant temperature domain above the superconducting critical temperature T_s . The same is valid for the crystal anisotropy, but we shall not present a thorough thermodynamic analysis of this problem. The crystal anisotropy effect can be considered for concrete systems with various crystal structures [6, 18]. Here we find enough to demonstrate that the anisotropy is not crucial for the description of the coexistence phase. Of course, our investigation confirms the general concept [18] that the anisotropy reduces the degree of degeneration of the ground state and, hence, stabilizes the ordering along the main crystal directions.

There exists a formal similarity between the phase diagram obtained in our investigation and the phase diagram of certain improper ferroelectrics [47, 48, 49, 50, 51, 52]. The variants of the theory of improper ferroelectrics, known before 1980, were criticized in Ref. [52] for their oversimplification and inconsistency with the experimental results. But the further development of the theory has no such disadvantage (see, e.g., Ref. [50, 51]). We use the advantage of the theory of improper ferroelectrics, where the concept of a “primary” order parameter triggered by a secondary order parameter (the electric polarization \mathbf{P}_e) has been initially introduced and exploited (see Ref. [50, 51, 52]). The mechanism of the M-triggered superconductivity in itinerant ferromagnets is formally identical to the mechanism of appearance of structural order triggered by the electric polarization \mathbf{P}_e in improper ferroelectrics (P -trigger). Recently, the effect of M -trigger has been used in a theoretical treatment of ferromagnetic Bose condensates [53].

1.5. Aims of the paper

In the remainder of this paper we shall consider the GL free energy functional of unconventional ferromagnetic superconductors. Our aim is to establish the uniform phases which are described by the GL free energy presented in Section 2.1. More information about the justification of this investigation is presented in Section 2.2. Note, as also mentioned in Section 2.2, that we investigate a quite general GL model in a situation of a lack of a concrete information about the values of the parameters of this model for concrete compounds (UGe₂, URhGe, ZrZn₂) where the ferromagnetic superconductivity has been discovered. On one side this lack of information makes impossible a detailed comparison of the theory to the available experimental data but on the other side our results are not bound to one or more concrete substances but can be applied to any unconventional ferromagnetic superconductor. In Section 3 we discuss the phases in nonmagnetic unconventional superconductors. In Section 4 the M-trigger effect will be described in the simple case of a single coupling (interaction) between the magnetization \mathbf{M} and the superconducting order parameter ψ in an isotropic model of ferromagnetic superconductors, where the anisotropy effects are ignored. In Section 5

the effect of another important coupling between the magnetization and the superconducting order parameter on the thermodynamics of the ferromagnetic superconductors is taken into account. In Section 6 the anisotropy effects are considered. In Section 7 we summarize and discuss our findings.

2. Ginzburg-Landau free energy

Following Refs. [18, 42, 43] in this Chapter we discuss the phenomenological theory of spin-triplet ferromagnetic superconductors and justify our consideration in Sections 3–6.

2.1. Model

Consider the GL free energy functional

$$F[\psi, \mathbf{M}] = \int d^3x f(\psi, \mathbf{M}), \quad (1)$$

where the free energy density $f(\psi, \mathbf{M})$ (for short hereafter called “free energy”) of a spin-triplet ferromagnetic superconductor is a sum of five terms:

$$f(\psi, \mathbf{M}) = f_S(\psi) + f'_F(\mathbf{M}) + f_I(\psi, \mathbf{M}) + \frac{\mathbf{B}^2}{8\pi} - \mathbf{B} \cdot \mathbf{M}. \quad (2)$$

In Eq. (2) $\psi = \{\psi_j; j = 1, 2, 3\}$ is the three-dimensional complex vector describing the superconducting order and $\mathbf{B} = (\mathbf{H} + 4\pi\mathbf{M}) = \nabla \times \mathbf{A}$ is the magnetic induction; \mathbf{H} is the external magnetic field, $\mathbf{A} = \{A_j; j = 1, 2, 3\}$ is the magnetic vector potential. The last two terms on the r.h.s. of Eq. (2) are related with the magnetic energy which includes both diamagnetic and paramagnetic effects in the superconductor (see, e.g., [32, 39, 54]).

In Eq. (2), the term $f_S(\psi)$ describes the superconductivity for $\mathbf{H} = \mathbf{M} \equiv 0$. This free energy part can be written in the form [18]

$$f_S(\psi) = f_{grad}(\psi) + a_s |\psi|^2 + \frac{b_s}{2} |\psi|^4 + \frac{u_s}{2} |\psi^2|^2 + \frac{v_s}{2} \sum_{j=1}^3 |\psi_j|^4, \quad (3)$$

with

$$\begin{aligned} f_{grad}(\psi) = & K_1 (D_i \psi_j)^* (D_i D_j) + K_2 [(D_i \psi_i)^* (D_j \psi_j) + (D_i \psi_j)^* (D_j \psi_i)] \\ & + K_3 (D_i \psi_i)^* (D_i \psi_i), \end{aligned} \quad (4)$$

where a summation over the indices i, j ($= 1, 2, 3$) is assumed and the symbol

$$D_j = -i\hbar \frac{\partial}{\partial x_j} + \frac{2|e|\hbar}{c} A_j \quad (5)$$

of covariant differentiation is introduced. In Eq. (3), $b_s > 0$ and $a_s = \alpha_s(T - T_s)$, where α_s is a positive material parameter and T_s is the critical temperature of a standard second order phase transition which may take place at $H = \mathcal{M} = 0$; $H = |\mathbf{H}|$, and

$\mathcal{M} = |\mathbf{M}|$. The parameter u_s describes the anisotropy of the spin-triplet Cooper pair whereas the crystal anisotropy is described by the parameter v_s [11, 18]. In Eq. (3) the parameters K_j , ($j = 1, 2, 3$) are related with the effective mass tensor of anisotropic Cooper pairs [18].

The term $f'_F(\mathbf{M})$ in Eq. (2) is the following part of the free energy of a standard isotropic ferromagnet:

$$f'_F(\mathbf{M}) = c_f \sum_{j=1}^3 |\nabla_j \mathbf{M}_j|^2 + a_f(T'_f) \mathbf{M}^2 + \frac{b_f}{2} \mathbf{M}^4 \quad (6)$$

where $\nabla_j = \partial/\partial x_j$, $b_f > 0$, and $a_f(T'_f) = \alpha_f(T - T'_f)$ is represented by the material parameter $\alpha_f > 0$ and the temperature T'_f ; the latter differs from the critical temperature T_f of the ferromagnet and this point will be discussed below. In fact, through Eq. (2) we have already added a negative term $(-2\pi\mathcal{M}^2)$ to the total free energy $f(\psi, \mathbf{M})$. This is obvious when we set $H = 0$ in Eq. (2). Then we obtain the negative energy $(-2\pi\mathcal{M}^2)$ which should be added to $f'_F(\mathbf{M})$. In this way one obtains the total free energy $f_F(\mathbf{M})$ of the ferromagnet in a zero external magnetic field, which is given by a modification of Eq. (6) according to the rule

$$f_F(a_f) = f'_F [a_f(T'_f) \rightarrow a_f(T_f)], \quad (7)$$

where $a_f = \alpha_f(T - T_f)$ and

$$T_f = T'_f + \frac{2\pi}{\alpha_f} \quad (8)$$

is the critical temperature of a standard ferromagnetic phase transition of second order. This scheme was used in studies of rare earth ternary compounds [32, 54, 55, 56]. Alternatively [57], one may work from the beginning with the total ferromagnetic free energy $f_F(a_f, \mathbf{M})$ as given by Eqs. (6) - (8) but in this case the magnetic energy included in the last two terms on the r.h.s. of Eq. (2) should be replaced with $H^2/8\pi$. Both ways of work are equivalent.

Finally, the term

$$f_I(\psi, \mathbf{M}) = i\gamma_0 \mathbf{M} \cdot (\psi \times \psi^*) + \delta \mathbf{M}^2 |\psi|^2. \quad (9)$$

in Eq. (2) describes the interaction between the ferromagnetic order parameter M and the superconducting order parameter ψ [42, 43]. The γ_0 -term is the most substantial for the description of experimentally found ferromagnetic superconductors [43] while the $\delta \mathbf{M}^2 |\psi|^2$ -term makes the model more realistic in the strong coupling limit because it gives the opportunity to enlarge the phase diagram including both positive and negative values of the parameter a_s . This allows for an extension of the domain of the stable ferromagnetic order up to zero temperatures for a wide range of values of the material parameters and the pressure P . Such a picture corresponds to the real situation in ferromagnetic compounds.

In Eq. (9) the coupling constant $\gamma_0 > 0$ can be represented in the form $\gamma_0 = 4\pi J$, where $J > 0$ is the ferromagnetic exchange parameter [43]. In general, the parameter

δ for ferromagnetic superconductors may take both positive and negative values. The values of the material parameters (T_s , T_f , α_s , α_f , b_s , u_s , v_s , b_f , K_j , γ_0 and δ) depend on the choice of the concrete substance and on intensive thermodynamic parameters, such as the temperature T and the pressure P .

2.2. Way of treatment

The total free energy (2) is a quite complex object of theoretical investigation. The various vortex and uniform phases described by this complex model cannot be investigated within a single calculation but rather one should focus on concrete problems. In Ref. [43] the vortex phase was discussed with the help of the criterion [58] for a stability of this state near the phase transition line $T_{c2}(H)$; see also, Ref. [59]. In case of $H = 0$ one should apply the same criterion with respect to the magnetization \mathcal{M} for small values of $|\psi|$ near the phase transition line $T_{c2}(\mathcal{M})$ as performed in Ref. [43]. Here we shall be interested in the uniform phases, namely, when the order parameters ψ and \mathbf{M} do not depend on the spatial vector $\mathbf{x} \in V$ (V is the volume of the superconductor). Thus our analysis will be restricted to the consideration of the coexistence of uniform (Meissner) phases and ferromagnetic order. We shall perform this investigation in details and, in particular, we shall show that the main properties of the uniform phases can be given within an approximation in which the crystal anisotropy is neglected. Moreover, some of the main features of the uniform phases in unconventional ferromagnetic superconductors can be reliably outlined when the Cooper pair anisotropy is neglected, too.

The assumption of a uniform magnetization \mathbf{M} is always reliable outside a quite close vicinity of the magnetic phase transition and under the condition that the superconducting order parameter ψ is also uniform, i.e. that vortex phases are not present at the respective temperature domain. This conditions are directly satisfied in type I superconductors but in type II superconductors the temperature should be sufficiently low and the external magnetic field should be zero. Moreover, the mentioned conditions for type II superconductors may turn insufficient for the appearance of uniform superconducting states in materials with quite high values of the spontaneous magnetization. In such cases the uniform (Meissner) superconductivity and, hence, the coexistence of this superconductivity with uniform ferromagnetic order may not appear even at zero temperature. Up to now type I unconventional ferromagnetic superconductors have not been yet found whereas the experimental data for the recently discovered compounds UGe₂, URhGe, and ZrZn₂ are not enough to conclude definitely either about the lack or the existence of uniform superconducting states at low and ultra-low temperatures.

In all cases, if real materials can be described by the general GL free energy (1) - (9), the ground state properties will be described by uniform states, which we shall investigate. The problem about the availability of such states in real materials at finite temperatures is quite subtle at the present stage of research when the experimental data are not enough. We shall assume that uniform phases may exist in some unconventional ferromagnetic superconductors. Moreover, we find convenient to emphasize that these

phases appear as solutions of the GL equations corresponding to the free energy (1) - (9). These arguments completely justify our study.

In case of a strong easy axis type of magnetic anisotropy, as is in UGe_2 [26], the overall complexity of mean-field analysis of the free energy $f(\psi, \mathbf{M})$ can be avoided by performing an “Ising-like” description: $\mathbf{M} = (0, 0, \mathcal{M})$, where $\mathcal{M} = \pm|\mathbf{M}|$ is the magnetization along the “ z -axis.” Further, because of the equivalence of the “up” and “down” physical states ($\pm\mathbf{M}$) the thermodynamic analysis can be performed within the “gauge” $\mathcal{M} \geq 0$. But this stage of consideration can also be achieved without the help of crystal anisotropy arguments. When the magnetic order has a continuous symmetry one may take advantage of the symmetry of the total free energy $f(\psi, \mathbf{M})$ and avoid the consideration of equivalent thermodynamic states that occur as a result of the respective symmetry breaking at the phase transition point but have no effect on thermodynamics of the system. In the isotropic system one may again choose a gauge, in which the magnetization vector has the same direction as z -axis ($|\mathbf{M}| = M_z = \mathcal{M}$) and this will not influence the generality of thermodynamic analysis. Here we shall prefer the alternative description within which the ferromagnetic state may appear through two equivalent “up” and “down” domains with magnetizations \mathcal{M} and $(-\mathcal{M})$, respectively.

We shall perform the mean-field analysis of the uniform phases and the possible phase transitions between such phases in a zero external magnetic field ($\mathbf{H} = 0$), when the crystal anisotropy is neglected ($v_s \equiv 0$). The only exception will be the consideration in Sec. 3, where we briefly discuss the nonmagnetic superconductors ($\mathcal{M} \neq 0$). For our aims we use notations in which the number of parameters is reduced. Introducing the parameter

$$b = (b_s + u_s + v_s) \quad (10)$$

we redefine the order parameters and the other parameters in the following way:

$$\begin{aligned} \varphi_j &= b^{1/4}\psi_j = \phi_j e^{\theta_j}, \quad M = b_f^{1/4}\mathcal{M}, \\ r &= \frac{a_s}{\sqrt{b}}, \quad t = \frac{a_f}{\sqrt{b_f}}, \quad w = \frac{u_s}{b}, \quad v = \frac{v_s}{b}, \\ \gamma &= \frac{\gamma_0}{b^{1/2}b_f^{1/4}}, \quad \gamma_1 = \frac{\delta}{(bb_f)^{1/2}}. \end{aligned} \quad (11)$$

Having in mind our approximation of uniform ψ and \mathbf{M} and the notations (10) - (11), the free energy density $f(\psi, M) = F(\psi, M)/V$ can be written in the form

$$\begin{aligned} f(\psi, M) &= r\phi^2 + \frac{1}{2}\phi^4 + 2\gamma\phi_1\phi_2M\sin(\theta_2 - \theta_1) + \gamma_1\phi^2M^2 + tM^2 + \frac{1}{2}M^4 \\ &\quad - 2w[\phi_1^2\phi_2^2\sin^2(\theta_2 - \theta_1) + \phi_1^2\phi_3^2\sin^2(\theta_1 - \theta_3) + \phi_2^2\phi_3^2\sin^2(\theta_2 - \theta_3)] \\ &\quad - v[\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_2^2\phi_3^2]. \end{aligned} \quad (12)$$

Note, that in this free energy the order parameters ψ and \mathbf{M} are defined per unit volume.

The equilibrium phases are obtained from the equations of state

$$\frac{\partial f(\mu_0)}{\partial \mu_\alpha} = 0, \quad (13)$$

where the series of symbols μ can be defined as, for example, $\mu = \{\mu_\alpha\} = (M, \phi_1, \dots, \phi_3, \theta_1, \dots, \theta_3)$; μ_0 denotes an equilibrium phase. The stability matrix \hat{F} of the phases μ_0 is defined by

$$\hat{F}(\mu_0) = \{F_{\alpha\beta}(\mu_0)\} = \frac{\partial^2 f(\mu_0)}{\partial \mu_\alpha \partial \mu_\beta}. \quad (14)$$

An alternative treatment can be done in terms of the real (ψ'_j) and imaginary (ψ''_j) parts of the complex numbers $\psi_j = \psi'_j + i\psi''_j$. The calculation with the moduli ϕ_j and phase angles θ_j of ψ_j has a minor disadvantage in cases of strongly degenerate phases when some of the angles θ_j remain unspecified. Then one should consistently use the properties of the respective broken continuous symmetry. Alternatively, one may do an alternative analysis with the help of the components ψ'_j and ψ''_j .

In order to avoid any ambiguity in our discussion let us note that we often use the term “existence” of a phase in order to indicate that it appears in experiments. This means that the phase, we consider, is either stable or metastable, in quite rare cases, when certain special experimental conditions allow the observation of metastable states in equilibrium. When a solution (phase) of Eq. (13) is obtained it is said that the respective phase “exists”, of course, under some “existence conditions” that are imposed on the parameters $\{\mu_\alpha\}$ of the theory. But this is just a registration of the fact that a concrete phase satisfies Eq. (13).

The problem about the thermodynamic stability of the phases that are solutions of Eq. (13) is solved with the help of the matrix (14) and, if necessary, with an additional analysis including the comparison of the free energies of phases which correspond to minima of the free energy in one and the same domain of parameters $\{\mu_\alpha\}$. Then the stable phase will be the phase that corresponds to a global minimum of the free energy. Therefore, when we discuss experimental situation in which some phase exists according to the experimental data, this means that it is a global minimum of the free energy, a fact determined by a comparison of free energies of the phases. If other minima of the free energy exist in a certain domain of parameters $\{\mu_\alpha\}$ then these minima are metastable equilibria, i.e. metastable phases. If a solution of Eq. (13) is not a minimum, it corresponds to an (absolutely) unstable equilibrium and the matrix (14) corresponding to this unstable phase is negatively definite.

When we determine the minima of the free energy by the requirement for a positive definiteness of the stability matrix (14), we are often faced with the problem of a “marginal” stability, i.e. the matrix is neither positively nor negatively definite. This is often a result of the degeneration of the states (phases) with broken continuous symmetry, and one should distinguish these cases. If the reason for the lack of a clear positive definiteness of the stability matrix is precisely the mentioned degeneration of

the ground state, one may reliably conclude that the respective phase is stable. If there is another reason, the analysis of the matrix (14) turns insufficient for our aims to determine the respective stability property. These cases are quite rare and happen for very particular values of the parameters $\{\mu_\alpha\}$.

3. Pure superconductivity

Let us set $M \equiv 0$ in Eq. (12) and briefly summarize the known results [18, 11] for the “pure superconducting case” when the magnetic order cannot appear and magnetic effects do not affect the stability of the normal and uniform (Meissner) superconducting phases. The possible phases can be classified by the structure of the complex vector order parameter $\psi = (\psi_1, \psi_2, \psi_3)$. We shall often use the moduli vector (ϕ_1, ϕ_2, ϕ_3) with magnitude $\phi = (\phi_1^2 + \phi_2^2 + \phi_3^2)^{1/2}$ but we must not forget the values of the phase angles θ_j .

The normal phase (0,0,0) is always a solution of the Eqs. (13). It is stable for $r \geq 0$, and corresponds to a free energy $f = 0$. Under certain conditions, six ordered phases [18, 11] occur for $r < 0$. Here we shall not repeat the detailed description of these phases [18, 11] but we shall briefly mention their structure.

The simplest ordered phase is of type $(\psi_1, 0, 0)$ with equivalent domains: $(0, \psi_2, 0)$ and $(0, 0, \psi_3)$. Multi-domain phases of more complex structure also occur, but we shall not always enumerate the possible domains. For example, the “two-dimensional” phases can be fully represented by domains of type $(\psi_1, \psi_2, 0)$ but there are also other two types of domains: $(\psi_1, 0, \psi_3)$ and $(0, \psi_2, \psi_3)$. As we consider the general case when the crystal anisotropy is present ($v \neq 0$), this type of phases possesses the property $|\psi_i| = |\psi_j|$.

The two-dimensional phases are two and have different free energies. To clarify this point let us consider, for example, the phase $(\psi_1, \psi_2, 0)$. The two complex numbers, ψ_1 and ψ_2 can be represented either as two-component real vectors, or, equivalently, as rotating vectors in the complex plane. One can easily show that Eq. (12) yields two phases: a collinear phase, when $(\theta_2 - \theta_1) = \pi k (k = 0, \pm 1, \dots)$, i.e. when the vectors ψ_1 and ψ_2 are collinear, and another (noncollinear) phase when the same vectors are perpendicular to each other: $(\theta_2 - \theta_1) = \pi(k + 1/2)$. Having in mind that $|\phi_1| = |\phi_2| = \phi/\sqrt{2}$, the domain $(\psi_1, \psi_2, 0)$ of the collinear phase is given by $(\pm 1, 1, 0)\phi/\sqrt{2}$, and the same domain for the noncollinear phase is given by $(\pm i, 1, 0)\phi/\sqrt{2}$. Similar representations can be given for the other two domains of these phases.

In addition to the mentioned three ordered phases, three other ordered phases exist. For these phases all three components ψ_j have nonzero equilibrium values. Two of them have equal to one another moduli ϕ_j , i.e., $\phi_1 = \phi_2 = \phi_3$. The third phase is of the type $\phi_1 = \phi_2 \neq \phi_3$ and is unstable so it cannot occur in real systems. The two three-dimensional phases with equal moduli of the order parameter components have different phase angles and, hence, different structure. The difference between any couple of angles θ_j is given by $\pm\pi/3$ or $\pm 2\pi/3$. The characteristic vectors of

this phase can be of the form $(e^{i\pi/3}, e^{-i\pi/3}, 1)\phi/\sqrt{3}$ and $(e^{2i\pi/3}, e^{-i2\pi/3}, 1)\phi/\sqrt{3}$. The second stable three dimensional phase is “real”, i.e. the components ψ_j lie on the real axis; $(\theta_j - \theta_j) = \pi k$ for any couple of angles θ_j and the characteristic vectors are $(\pm 1, \pm 1, 1)\phi/\sqrt{3}$. The stability properties of these five stable ordered phases were presented in details in Refs. [18, 11].

When the crystal anisotropy is not present ($v = 0$) the picture changes. The increase of the level of degeneracy of the ordered states leads to an instability of some phases and to a lack of some noncollinear phases. Both two- and three-dimensional real phases, where $(\theta_j - \theta_j) = \pi k$, are no more constrained by the condition $\phi_i = \phi_j$ but rather have the freedom of a variation of the moduli ϕ_j under the condition $\phi^2 = -r > 0$. The two-dimensional noncollinear phase exists but has a marginal stability [11]. All other noncollinear phases even in the presence of a crystal anisotropy ($v \neq 0$) either vanish or are unstable; for details, see Ref. [11]. This discussion demonstrates that the crystal anisotropy stabilizes the ordering along the main crystallographic directions, lowers the level of degeneracy of the ordered state related with the spontaneous breaking of the continuous symmetry and favors the appearance of noncollinear phases.

The crystal field effects related to the unconventional superconducting order were established for the first time in Ref. [18]. In our consideration of unconventional ferromagnetic superconductors in Sec. 4–7 we shall take advantage of these effects of the crystal anisotropy. In both cases $v = 0$ and $v \neq 0$ the matrix (14) indicates an instability of three-dimensional phases (all $\phi_j \neq 0$) with an arbitrary ratios ϕ_i/ϕ_j . As already mentioned, for $v \neq 0$ the phases of type $\phi_1 = \phi_2 \neq \phi_3$ are also unstable whereas for $v = 0$, even the phase $\phi_1 = \phi_2 = \phi_3 > 0$ is unstable.

4. Simple case of M-triggered superconductivity

Here we consider the Walker-Samokhin model [43] when only the $M\phi_1\phi_2$ -coupling between the order parameters ψ and M is taken into account ($\gamma > 0$, $\gamma_1 = 0$). Besides, we shall neglect the anisotropies ($w = v = 0$). The uniform phases and the phase diagram in this case were investigated in Refs. [44, 45, 46]. Here we summarize the main results in order to make a clear comparison with the new results presented in Sections 5 and 6. In this Section we set $\theta_3 \equiv 0$ and use the notation $\theta \equiv \Delta\theta = (\theta_2 - \theta_1)$. The symmetry of the system allows to introduce the notations without a loss of generality of the consideration.

4.1. Phases

The possible (stable, metastable and unstable) phases are given in Table 1 together with the respective existence and stability conditions. The stability conditions define the domain of the phase diagram where the respective phase is either stable or metastable [13]. The normal (disordered) phase, denoted in Table 1 by N always exists (for all temperatures $T \geq 0$) but is stable for $t > 0$, $r > 0$. The superconductivity phase denoted in Table 1 by SC1 is unstable. The same is valid for the phase of co-existence of ferromagnetism and superconductivity denoted in Table 1 by CO2. The

N-phase, the ferromagnetic phase (FM), the superconducting phases (SC1–3) and two of the phases of coexistence (CO1–3) are generic phases because they appear also in the decoupled case ($\gamma \equiv 0$). When the $M\phi_1\phi_2$ -coupling is not present, the phases SC1–3 are identical and represented by the order parameter ϕ where the components ϕ_j participate on equal footing. The asterisk attached to the stability condition of “the second superconductivity phase” (SC2), indicates that our analysis is insufficient to determine whether this phase corresponds to a minimum of the free energy. As we shall see later the phase SC2, as well as the other two purely superconducting phases and the coexistence phase CO1, have no chance to become stable for $\gamma \neq 0$. This is so, because the non-generic phase of coexistence of superconductivity and ferromagnetism (FS in Table 1), which does not exist for $\gamma = 0$ is stable and has a lower free energy in their domain of stability. Note, that a second domain ($M < 0$) of the FS phase exists and is denoted in Table 1 by FS*. Here we shall describe only the first domain (FS). The domain FS* is considered in the same way.

The cubic equation for M corresponding to FS (see Table 1) is shown in Fig. 1 for $\gamma = 1.2$ and $t = -0.2$. For any $\gamma > 0$ and t , the stable FS thermodynamic states are given by $r(M) < r_m = r(M_m)$ for $M > M_m > 0$, where M_m corresponds to the maximum of the function $r(M)$. Functions $M_m(t)$ and $M_0(t) = (-t + \gamma^2/2)^{1/2} = \sqrt{3}M_m(t)$ are drawn in Fig. 2 for $\gamma = 1.2$. Functions $r_m(t) = 4M_m^3(t)/\gamma$ for $t < \gamma^2/2$ (the line of circles in Fig. 3) and $r_e(t) = \gamma|t|^{1/2}$ for $t < 0$ define the borderlines of stability and existence of FS.

TABLE 1. Phases and their existence and stability properties [$\theta = (\theta_2 - \theta_1)$, $k = 0, \pm 1, \dots$].

Phase	order parameter	existence	stability domain
N	$\phi_j = M = 0$	always	$t > 0, r > 0$
FM	$\phi_j = 0, M^2 = -t$	$t < 0$	$r > 0, r^2 > \gamma^2 t$
SC1	$\phi_1 = M = 0, \phi^2 = -r$	$r < 0$	unstable
SC2	$\phi^2 = -r, \theta = \pi k, M = 0$	$r < 0$	$(t > 0)^*$
SC3	$\phi_1 = \phi_2 = M = 0, \phi_3^2 = -r$	$r < 0$	$r < 0, t > 0$
CO1	$\phi_1 = \phi_2 = 0, \phi_3^2 = -r, M^2 = -t$	$r < 0, t < 0$	$r < 0, t < 0$
CO2	$\phi_1 = 0, \phi^2 = -r, \theta = \theta_2 = \pi k, M^2 = -t$	$r < 0, t < 0$	unstable
FS	$2\phi_1^2 = 2\phi_2^2 = \phi^2 = -r + \gamma M, \phi_3 = 0$ $\theta = 2\pi(k - 1/4), \gamma r = (\gamma^2 - 2t)M - 2M^3$	$\gamma M > r$	$3M^2 > (-t + \gamma^2/2)$ $M > 0$
FS*	$2\phi_1^2 = 2\phi_2^2 = \phi^2 = -(r + \gamma M), \phi_3 = 0$ $\theta = 2\pi(k + 1/4), \gamma r = (2t - \gamma^2)M + 2M^3$	$-\gamma M > r$	$3M^2 > (-t + \gamma^2/2)$ $M < 0$

4.2. Phase diagram

We have outlined the domain in the (t, r) plane where the FS phase exists and is a minimum of the free energy. For $r < 0$ the cubic equation for M (see Table 1) and the existence and stability conditions are satisfied for any $M \geq 0$ provided $t \geq \gamma^2$. For $t < \gamma^2$ the condition $M \geq M_0$ have to be fulfilled, here the value $M_0 = (-t + \gamma^2/2)^{1/2}$ of M is obtained from $r(M_0) = 0$. Thus for $r = 0$ the N-phase is stable for $t \geq \gamma^2/2$, on

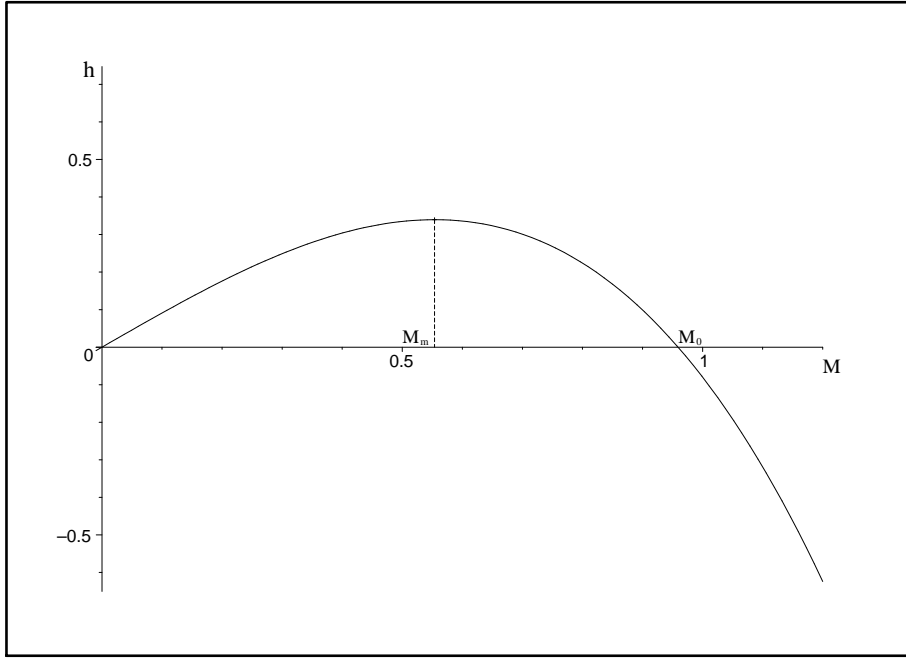


Figure 1: $h = \gamma r/2$ as a function of M for $\gamma = 1.2$, and $t = -0.2$.

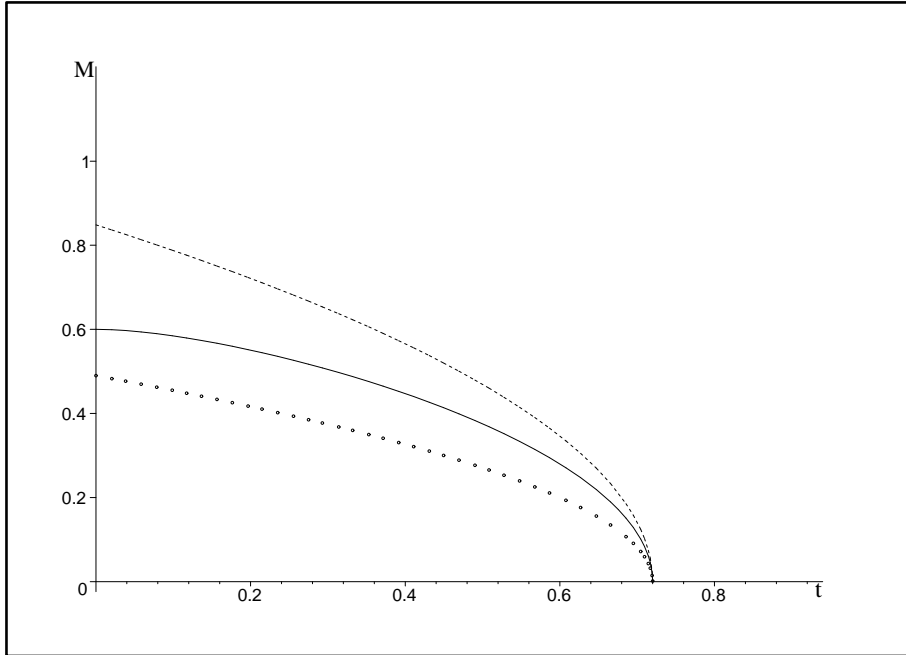


Figure 2: M versus t for $\gamma = 1.2$: the dashed line represents M_0 , the solid line represents M_{eq} , and the dotted line corresponds to M_m .

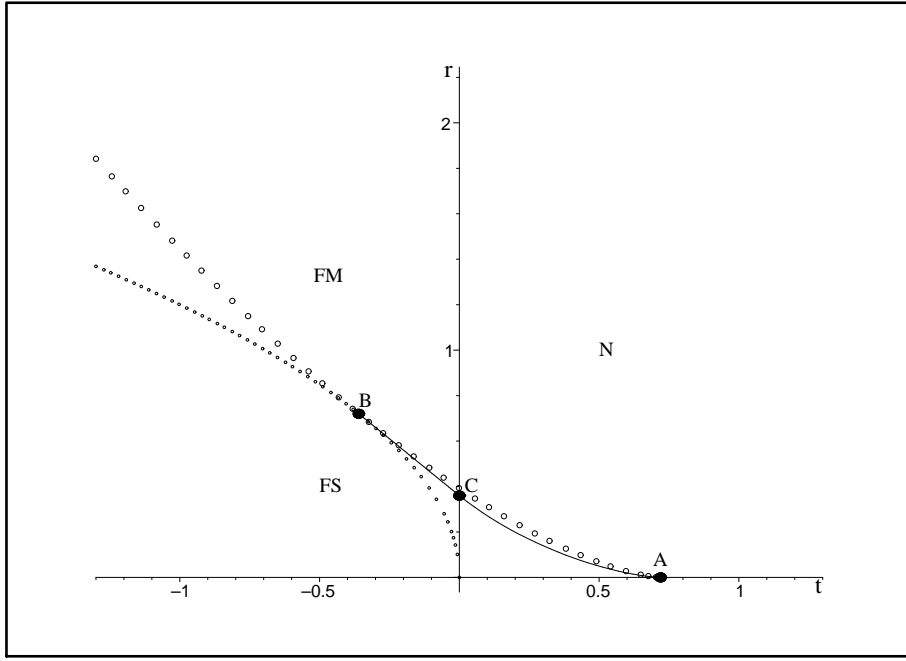


Figure 3: The phase diagram in the plane (t, r) with two tricritical points (A and B) and a triple point C; $\gamma = 1.2$. The domains of existence and stability of the phases N, FM and FS are shown.

the other hand FS is stable for $t \leq \gamma^2/2$. For $r > 0$, the requirement for the stability of FS leads to the inequalities

$$\max\left(\frac{r}{\gamma}, M_m\right) < M < M_0, \quad (15)$$

where $M_m = (M_0/\sqrt{3})$ and M_0 should be the positive solution of the cubic equation of state from Table 1; $M_m > 0$ gives a maximum of the function $r(M)$; see also Figs. 1 and 2.

The further analysis leads to the existence and stability domain of FS below the line AB given by circles (see Fig. 3). In Fig. 3 the curve of circles starts from the point A with coordinates $(\gamma^2/2, 0)$ and touches two other (solid and dotted) curves at the point B with coordinates $(-\gamma^2/4, \gamma^2/2)$. Line of circles represents the function $r(M_m) \equiv r_m(t)$ where

$$r_m(t) = \frac{4}{3\sqrt{3}\gamma} \left(\frac{\gamma^2}{2} - t\right)^{3/2}. \quad (16)$$

Dotted line is given by $r_e(t) = \gamma\sqrt{|t|}$. The inequality $r < r_m(t)$ is a condition for the stability of FS, whereas the inequality $r \leq r_e(t)$ for $(-t) \geq \gamma^2/4$ is a condition for the existence of FS as a solution of the respective equation of state. This existence condition for FS has been obtained from $\gamma M > r$ (see Table 1).

In the region on the left of the point B in Fig. 3, the FS phase satisfies the existence condition $\gamma M > r$ only below the dotted line. In the domain confined between the

lines of circles and the dotted line on the left of the point B the stability condition for FS is satisfied but the existence condition is broken. The inequality $r \geq r_e(t)$ is the stability condition of FM for $0 \leq (-t) \leq \gamma^2/4$. For $(-t) > \gamma^2/4$ the FM phase is stable for all $r \geq r_e(t)$.

In the region confined by the line of circles AB, the dotted line for $0 < (-t) < \gamma^2/4$, and the t -axis, the phases N, FS and FM have an overlap of stability domains. The same is valid for FS, the SC phases and CO1 in the third quadrant of the plane (t, r) . The comparison of the respective free energies for $r < 0$ shows that the stable phase is FS whereas the other phases are metastable within their domains of stability.

The part of the t -axis given by $r = 0$ and $t > \gamma^2/2$ is a phase transition line of second order which describes the N-FS transition. The same transition for $0 < t < \gamma^2/2$ is represented by the solid line AC which is the equilibrium transition line of a first order phase transition. This equilibrium transition curve is given by the function

$$r_{eq}(t) = \frac{1}{4} [3\gamma - (\gamma^2 + 16t))^{1/2}] M_{eq}(t), \quad (17)$$

where

$$M_{eq}(t) = \frac{1}{2\sqrt{2}} \left[\gamma^2 - 8t + \gamma (\gamma^2 + 16t)^{1/2} \right]^{1/2} \quad (18)$$

is the equilibrium value (jump) of the magnetization. The order of the N-FS transition changes at the tricritical point A.

The domain above the solid line AC and below the line of circles for $t > 0$ is the region of a possible overheating of FS. The domain of overcooling of the N-phase is confined by the solid line AC and the axes ($t > 0, r > 0$). At the triple point C with coordinates $[0, r_{eq}(0) = \gamma^2/4]$ the phases N, FM, and FS coexist. For $t < 0$ the straight line

$$r_{eq}^*(t) = \frac{\gamma^2}{4} + |t|, \quad -\gamma^2/4 < t < 0, \quad (19)$$

describes the extension of the equilibrium phase transition line of the N-FS first order transition to negative values of t . For $t < (-\gamma^2/4)$ the equilibrium phase transition FM-FS is of second order and is given by the dotted line on the left of the point B (the second tricritical point in this phase diagram). Along the first order transition line $r_{eq}^*(t)$ given by Eq. (19) the equilibrium value of M is $M_{eq} = \gamma/2$, which implies an equilibrium order parameter jump at the FM-FS transition equal to $(\gamma/2 - \sqrt{|t|})$. On the dotted line of the second order FM-FS transition the equilibrium value of M is equal to that of the FM phase ($M_{eq} = \sqrt{|t|}$). Note, that the FM phase does not exist below T_s and this seems to be a disadvantage of the model (12) with $\gamma_1 = 0$.

The equilibrium phase transition lines of the FM-FS and N-FS transition lines in Fig. 3 can be expressed by the respective equilibrium phase transition temperatures T_{eq} defined by the equations $r_e = r(T_{eq})$, $r_{eq} = r(T_{eq})$, $r_{eq}^* = r(T_{eq})$, and with the help of the relation $M_{eq} = M(T_{eq})$. This leads to some limitations on the possible variations of the parameters of the theory. For example, the critical temperature

($T_{eq} \equiv T_c$) of the FM-FS transition of second order ($\gamma^2/4 < -t$) is obtained in the form $T_c = (T_s + 4\pi J\mathcal{M}/\alpha_s)$, or, using $\mathcal{M} = (-a_f/\beta)^{1/2}$,

$$T_c = T_s - \frac{T^*}{2} + \left[\left(\frac{T^*}{2} \right)^2 + T^*(T_f - T_s) \right]^{1/2}, \quad (20)$$

where $T_f > T_s$, and $T^* = (4\pi J)^2 \alpha_f / \alpha_s^2 \beta$ is a characteristic temperature of the model (12) with $\gamma_1 = w = v = 0$. The investigation of the conditions for the validity of Eq. (20) leads to the conclusion that the FM-FS continuous phase transition (at $\gamma^2 < -t$) is possible only if the following condition is satisfied:

$$T_f - T_s > = (\varsigma + \sqrt{\varsigma}) T^*, \quad (21)$$

where $\varsigma = \beta \alpha_s^2 / 4b\alpha_f^2$. This means that the second order FM-FS transition should disappear for a sufficiently large γ -coupling. Such a condition does not exist for the first order transitions FM-FS and N-FS.

Taking into account the gradient term (4) in the free energy (2) should lead to a depression of the equilibrium transition temperature. As the magnetization increases with the decrease of the temperature, the vortex state should occur at temperatures which are lower than the equilibrium temperature T_{eq} of the homogeneous (Meissner) state. For example, the critical temperature (\tilde{T}_c) corresponding to the inhomogeneous (vortex) phase of FS-type has been evaluated [43] to be lower than the critical temperature (20): $(T_c - \tilde{T}_c) = 4\pi\mu_B\mathcal{M}/\alpha_s$ ($\mu_B = |e|\hbar/2mc$ - Bohr magneton). For $J \gg \mu_B$, we have $T_c \approx \tilde{T}_c$.

For $r > 0$, namely, for temperatures $T > T_s$ the superconductivity is triggered by the magnetic order through the γ -coupling. The superconducting phase for $T > T_s$ is entirely in the (t, r) domain of the ferromagnetic phase. Therefore, the uniform superconducting phase can occur for $T > T_s$ only through a coexistence with the ferromagnetic order.

In the next Sections we shall focus on the temperature range $T > T_s$ which seems to be of main practical interest. We shall not dwell on the superconductivity in the fourth quadrant ($t > 0, r < 0$) of the (t, r) diagram where pure superconductivity phases are possible in systems with $T_s > T_f$ (this is not the case for UGe₂, URhGe, and ZrZn₂). Besides, we shall not discuss the possible metastable phases in the third quadrant ($t < 0, r < 0$) of the (t, r) diagram.

4.3. Magnetic susceptibility

Consider the longitudinal magnetic susceptibility $\chi_1 = (\chi_V/V)$ per unit volume [46]. The external magnetic field $\mathbf{H} = (0, 0, H)$ with $H = (\partial f / \partial \mathcal{M})$ has the same direction as the magnetization \mathbf{M} . We shall calculate the quantity $\chi = \sqrt{\beta_f} \chi_1$ for the equilibrium thermodynamic states μ_0 given by Eq. (13). Having in mind the relations (11)

between M and \mathcal{M} , and between ψ and φ we can write

$$\chi^{-1} = \frac{d}{dM_0} \left[\left(\frac{\partial f}{\partial M} \right)_{T, \varphi_j} \right]_{\mu_0}, \quad (22)$$

where the equilibrium magnetization M_0 and equilibrium superconducting order parameter components φ_{0j} should be taken for the respective equilibrium phase (see Table 1, where the suffix “0” of ϕ , θ , and M has been omitted; hereafter the same suffix will be often omitted, too). Note that the value of the equilibrium magnetization M in FS is the maximal nonnegative root of the cubic equation in M given in Table 1.

Using Eq. (22) we obtain the susceptibility χ of the FS phase in the form

$$\chi^{-1} = -\gamma^2 + 2t + 6M^2. \quad (23)$$

The susceptibility of the other phases has the usual expression

$$\chi^{-1} = 2t + 6M^2. \quad (24)$$

Eq. (24) yields the known results for the paramagnetic susceptibility ($\chi_P = 1/2t$; $t > 0$), corresponding to the normal phase, and for the ferromagnetic susceptibility ($\chi_F = 1/4|t|$; $t < 0$), corresponding to FM. These susceptibilities can be compared with the susceptibility χ of FS. As the susceptibility χ of FS cannot be analytically calculated for the whole domain of stability of FS, we shall consider the close vicinity of the N-FS and FM-FS phase transition lines.

Near the second order phase transition line on the left of the point B ($t < -\gamma^2/4$), the magnetization has a smooth behaviour and the magnetic susceptibility does not exhibit any singularities (jump or divergence). For $t > \gamma^2/2$, the magnetization is given by $M = (s_- + s_+)$, where

$$s_{\pm} = \left\{ -\frac{\gamma r}{4} \pm \left[\frac{(t - \gamma^2/2)^3}{27} + \left(\frac{\gamma r}{4} \right)^2 \right]^{1/2} \right\}^{1/3}. \quad (25)$$

For $r = 0$, $M = 0$, whereas for $|\gamma r| \ll (t - \gamma^2/2)$ and $r = 0$ one may obtain $M \approx -\gamma r/(2t - \gamma^2) \ll 2t$. This means that in a close vicinity ($r < 0$) of $r = 0$ along the second order phase transition line ($r = 0, t > \gamma^2$) the magnetic susceptibility is well described by the paramagnetic law $\chi_P = (1/2t)$. For $r < 0$ and $t \rightarrow \gamma^2/2$, we obtain $M = -(\gamma r/2)^{1/3}$ which yields

$$\chi^{-1} = 6 \left(\frac{\gamma |r|}{2} \right)^{2/3}. \quad (26)$$

On the phase transition line AC we have

$$M_{eq}(t) = \frac{1}{2\sqrt{2}} \left[\gamma^2 - 8t + \gamma (\gamma^2 + 16t)^{1/2} \right]^{1/2} \quad (27)$$

and, hence,

$$\chi^{-1} = -4t - \frac{\gamma^2}{4} \left[1 - 3 \left(1 + \frac{16t}{\gamma^2} \right)^{1/2} \right]. \quad (28)$$

At the tricritical point A this result yields $\chi^{-1}(A) = 0$, whereas at the triple point C with coordinates $(0, \gamma^2/4)$ we have $\chi(C) = (2/\gamma^2)$. On the line BC we obtain $M = \gamma/2$ and, hence,

$$\chi^{-1} = 2t + \frac{\gamma^2}{2}. \quad (29)$$

At the tricritical point B with coordinates $(-\gamma^2/4, \gamma^2/2)$ this result yields $\chi^{-1}(B) = 0$.

In order to investigate the magnetic susceptibility tensor we shall slightly extend the framework of our treatment by considering arbitrary orientations of the vectors \mathbf{H} and \mathbf{M} . We shall denote the spatial directions $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ as $(1, 2, 3)$.

The components of the inverse magnetic susceptibility tensor

$$\hat{\chi}_1^{-1} = \hat{\chi}^{-1} \sqrt{b_f} = \{\chi_{ij}^{-1}\} \sqrt{b_f} \quad (30)$$

can be represented in the form

$$\chi_{ij}^{-1} = 2(t + M^2)\delta_{ij} + 4M_i M_j + i\gamma \frac{\partial}{\partial M_j} (\varphi \times \varphi^*)_i, \quad (31)$$

where M and φ_j are to be taken at their equilibrium values: $M_0, \varphi_{0j}, \theta_{0j}$. The last term in the r.h.s. of Eq. (28) is equal to zero for all phases in Table 1 except for FS (and FS*). When the last term in Eq. (29) is equal to zero we obtain the known result the susceptibility tensor for second order phase transitions (see, e.g., Ref. [13]).

Consider the FS phase, where ϕ_j depends on M_j . Now we can choose again $\mathbf{M} = (0, 0, M)$ and use our results for the equilibrium values of ϕ_j, θ and M (see Table 1). Then the components χ_{ij}^{-1} corresponding to FS are given by

$$\chi_{ij}^{-1} = 2(t + M^2)\delta_{ij} + 4M_i M_j - \gamma^2 \delta_{i3}. \quad (32)$$

Thus we have $\chi_{i \neq j}^{-1} = 0$,

$$\chi_{11}^{-1} = \chi_{22}^{-1} = 2(t + M^2), \quad (33)$$

and χ_{33}^{-1} coincides with the inverse longitudinal susceptibility χ^{-1} given by Eq. (23).

4.4. Entropy and specific heat

The entropy $S(T) \equiv (\tilde{S}/V) = -V\partial(f/\partial T)$ and the specific heat $C(T) \equiv (\tilde{C}/V) = T(\partial S/\partial T)$ per unit volume V are calculated in a standard way [13]. We are interested in the jumps of these quantities on the N-FM, FM-FS, and N-FS transition lines. The behaviour of $S(T)$ and $C(T)$ near the N-FM phase transition and near the FM-FS phase transition line of second order on the left of the point B (Fig. 3) is known from the standard theory of critical phenomena (see, e.g., Ref. [13] and for this reason we focus our attention on the phase transitions of type FS-FM and FS-N for $(t > -\gamma^2/4)$, i.e., on the right of the point B in Fig. 3.

Using the equations for the order parameters ψ and M (Table 1) and applying the standard procedure for the calculation of S , we obtain the general expression

$$S(T) = -\frac{\alpha_s}{\sqrt{b}}\phi^2 - \frac{\alpha_f}{\sqrt{\beta}}M^2. \quad (34)$$

The next step is to calculate the entropies $S_{FS}(T)$ and S_{FM} of the ordered phases FS and FM. Note, that use the usual convention $F_N = V f_N = 0$ for the free energy of the N-phase and, hence, we must set $S_N(T) = 0$.

Consider the second order phase transition line ($r = 0$, $t > \gamma^2/2$). Near this line $S_{FS}(T)$ is a smooth function of T and has no jump but the specific heat C_{FS} has a jump at $T = T_s$, i.e. for $r = 0$. This jump is given by

$$\Delta C_{FS}(T_s) = \frac{\alpha_s^2 T_s}{b} \left[1 - \frac{1}{1 - 2t(T_s)/\gamma^2} \right]. \quad (35)$$

The jump $\Delta C_{FS}(T_s)$ is higher than the usual jump $\Delta C(T_c) = T_c \alpha^2/b$ known from the Landau theory of standard second order phase transitions [13].

The entropy jump $\Delta S_{AC}(T) \equiv S_{FS}(T)$ on the line AC is obtained in the form

$$\Delta S_{AC}(T) = -M_{eq} \left\{ \frac{\alpha_s \gamma}{4\sqrt{b}} \left[1 + \left(1 + \frac{16t}{\gamma^2} \right)^{1/2} \right] - \frac{\alpha_f}{\sqrt{\beta}} M_{eq} \right\}, \quad (36)$$

where M_{eq} is given by Eq. (18). From Eqs. (18) and (36), we have $\Delta S(t = \gamma^2/2) = 0$, i.e., $\Delta S(T)$ becomes equal to zero at the tricritical point A . Besides we find from Eqs. (18) and (36) that at the triple point C the entropy jump is given by

$$\Delta S(t = 0) = -\frac{\gamma^2}{4} \left(\frac{\alpha_s}{\sqrt{b}} + \frac{\alpha_f}{\sqrt{\beta}} \right). \quad (37)$$

On the line BC the entropy jump is defined by $\Delta S_{BC}(T) = [S_{FS}(T) - S_{FM}(T)]$. We obtain

$$\Delta S_{BC}(T) = \left(|t| - \frac{\gamma^2}{4} \right) \left(\frac{\alpha_s}{\sqrt{b}} + \frac{\alpha_f}{\sqrt{\beta}} \right). \quad (38)$$

At the tricritical point B this jump is equal to zero as it should be. The calculation of the specific heat jump on the first order phase transition lines AC and BC is redundant for two reasons. Firstly, the jump of the specific heat at a first order phase transition differs from the entropy by a factor of order of unity. Secondly, in caloric experiments where the relevant quantity is the latent heat $Q = T\Delta S(T)$, the specific heat jump can hardly be distinguished.

4.5. Note about a simplified theory

The consideration in this Section as well as in Sections 5 and 6 can be performed within an approximate scheme, known from the theory of improper ferroelectrics (see, e.g., Ref. [52]). The idea of the approximation is in the supposition that the order parameter

M is small enough so that one can neglect M^4 -term in the free energy. Within this approximation one easily obtains from the data for FS presented in Table 1 or by a direct calculation of the respective reduced free energy that the order parameters ϕ and M of FS are described by the simple equalities $r = (\gamma M - \phi^2)$ and $M = (\gamma/2t)\phi^2$. Of course, one may perform this simple analysis from the very beginning. For ferroelectrics this approximation gives a substantial departure of theory from experiment [52]. In general, the domain of reliability of such an approximation should be the close vicinity of the ferromagnetic phase transition, i.e. temperatures near to the critical temperature T_f . On the other hand, this discussion is worthwhile only if the “primary” order parameter also exists in the same (narrow) temperature domain ($\phi > 0$). Therefore this approximation has some application in systems, where $T_s \geq T_f$.

For $T_s < T_f$, one may simplify our thorough analysis by a supposition for a relatively small value of the modulus ϕ of the superconducting order parameter. This approximation should be valid in some narrow temperature domain near the line of second order phase transition from FM to FS.

5. Effect of symmetry conserving coupling

Here we consider the case when both coupling parameters γ and γ_1 are different from zero. In this way we shall investigate the effect of the symmetry conserving γ_1 -term in the free energy on the thermodynamics of the system. Note that when γ is equal to zero the analysis is quite easy and the results are known from the theory of bicritical and tetracritical points [13, 50, 60, 61]. For the problem of coexistence of conventional superconductivity and ferromagnetic order this analysis ($\gamma = 0, \gamma_1 \neq 0$) was made in Ref. [32]. Once again we postpone the consideration of anisotropy effects by setting $w = v = 0$. The present analysis is much more difficult than that in Sec. 4, and cannot be performed only by analytical calculations; rather, some complementary numerical analysis is needed. Our investigation is based to a great extent on analytical calculations but a numerical analysis has been also performed in order to obtain concrete conclusions.

5.1. Phases

The calculations show that for temperatures $T > T_s$, i.e., for $r > 0$, we have three stable phases. Two of them are quite simple: the normal (N -) phase with existence and stability domains shown in Table 1, and the FM phase with the existence condition $t < 0$ as shown in Table 1, and a stability domain defined by the inequalities $r > \gamma_1 t$ and

$$r > \gamma_1 t + \gamma \sqrt{-t}. \quad (39)$$

The third stable phase for $r > 0$ is a more complex variant of the mixed phase FS and its domain FS*, discussed in Section 4. The symmetry of the FS phase coincides with that found in [43]

Let us also mention that for $r < 0$ five pure superconducting ($M = 0, \phi > 0$) phases exist. Two of these phases, $(\phi_1 > 0, \phi_2 = \phi_3 = 0)$ and $(\phi_1 = 0, \phi_2 > 0, \phi_3 > 0)$ are

unstable. Two other phases, $(\phi_1 > 0, \phi_2 > 0, \phi_3 = 0, \theta_2 = \theta_1 + \pi k)$ and $(\phi_1 > 0, \phi_2 > 0, \phi_3 > 0, \theta_2 = \theta_1 + \pi k, \theta_3 - \text{arbitrary}; k = 0, \pm 1, \dots)$ show a marginal stability for $t > \gamma_1 r$.

Only one of the five pure superconducting phases, namely, the phase SC3, given in Table 1, is stable. In the present case of $\gamma_1 \neq 0$ the values of ϕ_j and the existence domain of SC3 are the same as shown in Table 1 for $\gamma_1 = 0$ but the stability domain is different and is given by $t > \gamma_1 r$. When the anisotropy effects are taken into account the phases exhibiting marginal stability within the present treatment may receive a further stabilization. Besides, three other mixed phases ($M \neq 0, \phi > 0$) exist for $r < 0$ but one of them is metastable (for $\gamma_1^2 > 1, t < \gamma_1 r$, and $r < \gamma_1 t$) and the other two are absolutely unstable. Here the thermodynamic behaviour for $r < 0$ is much more abundant in phases than in the case of improper ferroelectrics with two component primary order parameter [50]. However, at this stage of experimental needs about the properties of unconventional ferromagnetic superconductors the investigation of the phases for temperatures $T < T_s$ is not of primary interest and for this reason we shall focus on the relatively higher temperature domain $r > 0$.

The FS phase is described by the following equations:

$$\phi_1 = \phi_2 = \frac{\phi}{\sqrt{2}}, \quad \phi_3 = 0, \quad (40)$$

$$\phi^2 = (\pm \gamma M - r - \gamma_1 M^2), \quad (41)$$

$$(1 - \gamma_1^2)M^3 \pm \frac{3}{2}\gamma\gamma_1 M^2 + \left(t - \frac{\gamma^2}{2} - \gamma_1 r\right)M \pm \frac{\gamma r}{2} = 0, \quad (42)$$

and

$$(\theta_2 - \theta_1) = \mp \frac{\pi}{2} + 2\pi k, \quad (43)$$

($k = 0, \pm 1, \dots$). The upper sign in Eqs. (41) - (43) corresponds to the FS domain in which $\sin(\theta_2 - \theta_1) = -1$ and the lower sign corresponds to the FS* domain. Here we have a generalization of the two-domain phase FS discussed in Section 4 and for this reason we use the same notations. The analysis of the stability matrix (14) for these phase domains shows that FS is stable for $M > 0$ and FS* is stable for $M < 0$, just like our result in Section 4. As these domains belong to the same phase, namely, have the same free energy and are thermodynamically equivalent, we shall consider one of them, for example, FS. Besides, our analysis of Eqs. (40) - (43) shows that FS exists and is stable in a broad domain of the (t, r) diagram, including substantial regions corresponding to $r > 0$.

5.2. Phase stability and phase diagram

In order to outline the phase diagram (t, r) we shall use the information given above for the other three phases which have their own domains of stability in the (t, r) plane: N, FM, and FS. The phase diagram for concrete parameters of γ and γ_1 is shown in

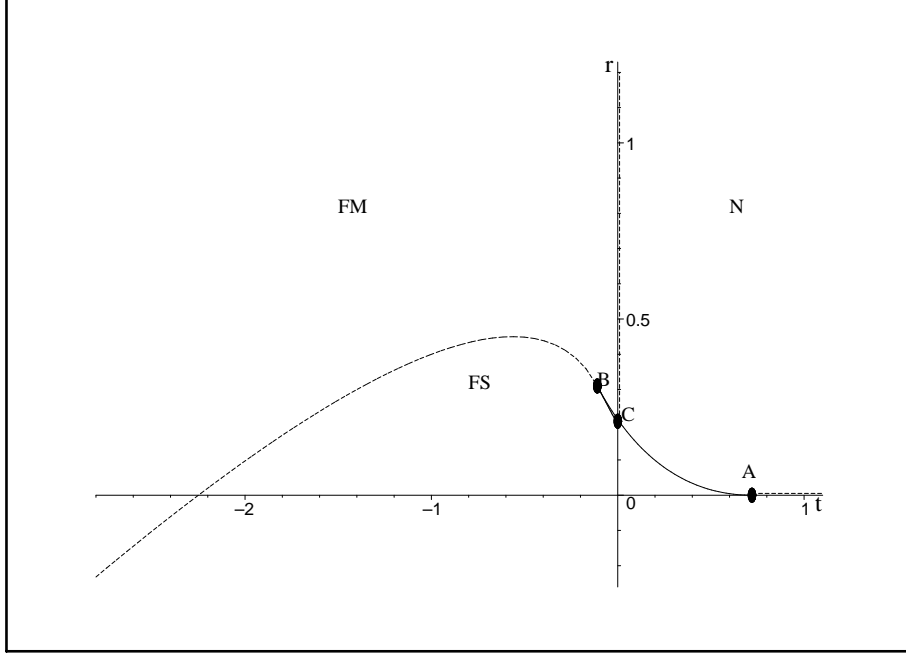


Figure 4: The phase diagram in the (t, r) plane for $\gamma = 1.2$, $\gamma_1 = 0.8$ and $w = 0$.

Fig. 4. The phase transition between the normal and FS phases is of first order and goes along the equilibrium line AC. It is given by the equation:

$$r_{eq}(t) = \frac{M_{eq}}{(\gamma_1 M_{eq} - \gamma/2)} \left[(1 - \gamma_1^2) M_{eq}^2 + \frac{3}{2} \gamma \gamma_1 M_{eq} + (t - \frac{\gamma^2}{2}) \right]. \quad (44)$$

The equilibrium value M_{eq} on the line AC is found by setting the equilibrium free energy $f_{FS}(\mu_0)$ of FS equal to zero, i.e. equal to the free energy ($f_N = 0$) of the N-phase. We have obtained the equilibrium energy f_N as a function of the magnetization:

$$f_{FS} = -\frac{M^2}{2(M\gamma_1 - \gamma/2)^2} \times \left\{ (1 - \gamma_1^2) M^4 + \gamma \gamma_1 M^3 + 2 \left[t(1 - \gamma_1^2) - \frac{\gamma^2}{8} \right] M^2 - 2\gamma \gamma_1 t M + t(t - \frac{\gamma^2}{2}) \right\}, \quad (45)$$

where $M \equiv M_{eq}$ (hereafter the suffix “eq” will be often omitted).

The numerical analysis of the free energy (45) as a polynomial of M shows that the expression in the curly brackets has one positive zero in the interval of values of t from $t = \gamma^2/2$ (point A in Fig. 4) up to $t = 0$, where $M_{t=0} = \gamma/2(\gamma_1 + 1)$. As far as the obtained values for M are in the interval $0 \leq M < (\gamma/2\gamma_1)$ the existence condition of FS, namely,

$$\phi^2 = \frac{M(M^2 + t)}{(\gamma/2 - \gamma_1 M)} \geq 0, \quad (46)$$

is also satisfied.

At the triple point C with coordinates $t = 0$, $r = \gamma^2/4(\gamma_1 + 1)$ three phases (N, FM, and FS) coexist. To find the magnetization M on the equilibrium curve BC of the first order phase transition FM-FS for $t < 0$, we use the equality $f_{FM} = f_{FS}$, or, equivalently,

$$\frac{(M^2 + t^2)^2}{2(M\gamma_1 - \gamma/2)^2} \left[\frac{\gamma}{2} - M(1 + \gamma_1) \right] \left[\frac{\gamma}{2} + M(1 - \gamma_1) \right] = 0. \quad (47)$$

Then the function $r_{eq}(t)$ for $t < 0$ will have the form

$$r_{eq}(t) = \frac{\gamma^2}{4(1 + \gamma_1)} - t, \quad (48)$$

This function describes the line BC of first order phase transition (see Fig. 4) which terminates at the tricritical point B with coordinates

$$t_B = -\frac{\gamma^2}{4(1 + \gamma_1)^2}, \quad r_B = \frac{\gamma^2(2 + \gamma_1)}{4(1 + \gamma_1)^2}. \quad (49)$$

To the left of the tricritical point the second order phase transition curve is given by the relation,

$$r_e(t) = \gamma\sqrt{-t} + \gamma_1 t, \quad (50)$$

which coincides with the stability condition (39) of FM. This line intersects t-axis for $t = (-\gamma^2/\gamma_1^2)$ and is well defined also for $r < 0$. On the curve $r_e(t)$ the magnetization is $M = \sqrt{-t}$ and the superconducting order parameter is equal to zero ($\phi = 0$). The function $r_e(t)$ has a maximum at the point $(t, r) = (-\gamma^2/4\gamma_1^2, \gamma^2/4\gamma_1)$; here $M = (\gamma/2\gamma_1)$. When this point is approached the second derivative of the free energy with respect to M tends to infinity, but as we shall see later the inclusion of the anisotropy of triplet pairing smears this singularity. The result for the curves $r_{FS}(t)$ of equilibrium phase transitions (N-FS and FM-FS) can be used to define the respective equilibrium phase transition temperatures T_{FS} .

We shall not discuss the region, $t > 0$, $r < 0$, because we have supposed from the very beginning of our analysis that the transition temperature for the ferromagnetic order T_f is higher than the superconducting transition temperature T_s , as it is for the known unconventional ferromagnetic superconductors. But this case may become of substantial interest when, as one may expect, materials with $T_f < T_s$ will be discovered.

The stability conditions of FS can be written in the general form

$$\frac{-M^2 + \gamma\gamma_1 M - t - \gamma^2/2}{M\gamma_1 - \gamma/2} \geq 0, \quad (51)$$

$$\gamma M \geq 0, \quad (52)$$

$$\frac{1}{M\gamma_1 - \gamma/2} \left[\gamma_1(1 - \gamma_1^2)M^3 - \frac{3}{4}\gamma(1 - 2\gamma_1^2)M^2 - \frac{3}{4}\gamma^2\gamma_1 M - \frac{\gamma}{4}(t - \gamma^2/2) \right] \geq 0. \quad (53)$$

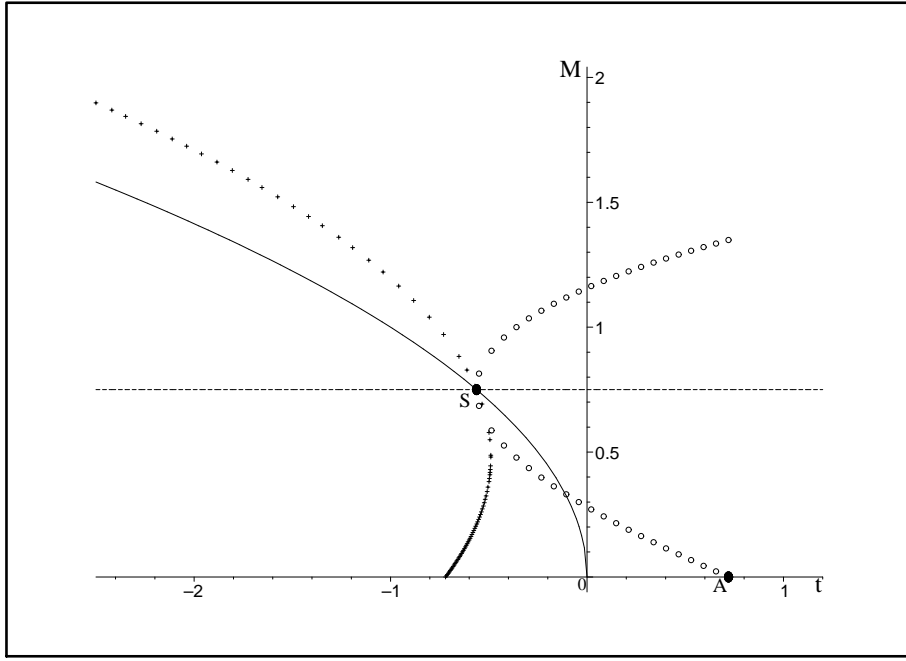


Figure 5: The dependence $M(t)$ as an illustration of stability analysis for $\gamma = 1.2$ and $\gamma_1 = 0.8$.

Our consideration of the stability conditions (51) - (53) together with the existence condition Eq. (46) of the phase FS is illustrated by the picture shown in Fig. 5.

For $0 \leq t \leq \gamma^2/2$ and $0 < M < (\gamma/2\gamma_1)$ conditions (46) and (51) are satisfied. Condition (53) is a cubic equation in $M(t)$ which for the above values of the parameter t has three real roots, one of them negative. The positive roots, $M(t) > 0$, as function of t are drawn by circles in Fig. 5 and it is obvious that the condition (53) will be satisfied for those values of $M(t)$ that are between the two circled curves. The smaller positive root of Eq. (53) intersects t -axis for $t = \gamma^2/2$ (point A in Fig. 5). Note, that $M = \gamma/(2\gamma_1)$ is given by the horizontal dashed line. For $t \leq -\gamma^2(2 - \gamma_1^2)/4$ the stability condition (51) has two real roots shown by curves with crosses in Fig. 5. For negative values of the parameter t we shall consider also the curve $M = \sqrt{-t}$ which is the solution of existence condition (46) and is depicted by solid line in Fig. 5. For $(-\gamma^2/4\gamma_1^2) < t < 0$ the FS phase exists and is stable when $\gamma/(2\gamma_1) \geq M \geq \sqrt{-t}$.

The point S in Fig. 5 with coordinates $(-\gamma^2/(4\gamma_1^2), \gamma/(2\gamma_1))$ is singular in sense that l.h.s. of conditions (51) and (53) go to infinity there. When $t > (-\gamma^2/4\gamma_1^2)$ the existence condition (46) implies $\gamma/(2\gamma_1) < M < \sqrt{-t}$. The stability condition (53) is always satisfied (two complex conjugate roots and one negative root) and condition (51) will be fulfilled for values of M between the two curves denoted by crosses in Fig. 5.

5.3. Discussion

The shape of the equilibrium phase transition lines corresponding to the phase transitions N-SC, N-FS, and FM-FS is similar to that for the simpler case $\gamma_1 = 0$ and we

shall not dwell on the variation of the size of the phase domains with the variations of the parameter γ_1 from zero to values constrained by the condition $\gamma_1^2 < 1$. Besides one may generalize our treatment (Section 4) of the magnetic susceptibility tensor and the thermal quantities in this more complex case and to demonstrate the dependence of these quantities on γ_1 . We shall not dwell on these problems. But an important qualitative difference between the equilibrium phase transition lines shown in Figs. 1 and 4 cannot be omitted. The second order phase transition line $r_e(t)$, shown by the dotted line on the left of point B in Fig. 1, tends to large positive values of r for large negative values of t and remains in the “second quadrant” ($t < 0, r > 0$) of the plane (t, r) while the respective second order phase transition line in Fig. 4 crosses the t -axis in the point $t = -\gamma^2/\gamma_1^2$ and is located in the third quadrant ($t < 0, r < 0$) for all possible values $t < -\gamma^2/\gamma_1^2$. This means that the ground state (at 0 K) of systems with $\gamma_1 = 0$ will be always the FS phase whereas two types of ground states, FM and FS, are allowed for systems with $0 < \gamma_1^2 < 1$. The latter seems more realistic in view of comparison of theory and experiment, especially, in ferromagnetic compounds like UGe₂, URhGe, and ZrZn₂. The neglecting of the γ_1 -term does not allow to describe the experimentally observed presence of FM phase at quite low temperatures and relatively low pressure P .

The final aim of the phase diagram investigation is the outline of the (T, P) diagram. Important conclusions about the shape of the (T, P) diagram can be made from the form of the (t, r) diagram without an additional information about the values of the relevant material parameters (a_s, a_f, \dots) and their dependence on the pressure P . One should know also the characteristic temperature T_s , which has a lower value than the experimentally observed [26, 27, 28, 30, 31] phase transition temperature ($T_{FS} \sim 1K$) to the mixed (FS) phase. A supposition about the dependence of the parameters a_s and a_f on the pressure P was made in Ref. [43]. Our results for $T_f \gg T_s$ show that the phase transition temperature T_{FS} varies with the variation of the system parameters ($\alpha_s, \alpha_f, \dots$) from values which are much higher than the characteristic temperature T_s up to zero temperature. This is seen from Fig. 4.

6. Anisotropy effects

When the anisotropy of the Cooper pairs is taken in consideration, there will be not drastic changes in the shape the phase diagram for $r > 0$ and the order of the respective phase transitions. Of course, there will be some changes in the size of the phase domains and the formulae for the thermodynamic quantities. The parameter w will also insert a slight change in the values of the thermodynamic quantities like the magnetic susceptibility and the entropy and specific heat jumps at the phase transition points.

Besides, and this seems to be the main anisotropy effect, the w - and v -terms in the free energy lead to a stabilization of the order along the main crystal directions which, in other words, means that the degeneration of the possible ground states (FM, SC, and FS) is considerably reduced. This means also a smaller number of marginally stable states which are encountered by the analysis of the definiteness of the stability matrix

(14). All anisotropy effects can be verified by the investigation of the free energy (12) which includes the w - and v -terms.

We have made the above general conclusions on the basis of a detailed analysis of the effect of the Cooper pair anisotropy (w -) term, as well as on the basis of a preliminary analysis of the total free energy (12), where the crystal anisotropy (v -) term is also taken into account. Here we shall present our basic results for the effect of the Cooper pair anisotropy on the FS phase; the crystal anisotropy is neglected ($v = 0$).

The dimensionless anisotropy parameter $w = \bar{u}/(u + \bar{u})$ can be either positive or negative depending on the sign of \bar{u} . Obviously when $\bar{u} > 0$, the parameter w will be positive too ($0 < w < 1$). We shall illustrate the influence of Cooper-pair anisotropy in this case. The order parameters (M, ϕ_j, θ_j) are given by Eqs. (40), (43),

$$\phi^2 = \frac{\pm \gamma M - r - \gamma_1 M^2}{(1 - w)} \geq 0, \quad (54)$$

and

$$(1 - w - \gamma_1^2)M^3 \pm \frac{3}{2}\gamma\gamma_1 M^2 + \left[t(1 - w) - \frac{\gamma^2}{2} - \gamma_1 r\right]M \pm \frac{\gamma r}{2} = 0, \quad (55)$$

where the meaning of the upper and lower sign is the same as explained just below Eq. (43). We consider the FS domain corresponding to the upper sign in the Eq. (54) and (55). The stability conditions for FS read,

$$\frac{(2 - w)\gamma M - r - \gamma_1 M^2}{1 - w} \geq 0, \quad (56)$$

$$\frac{1 - 2w}{1 - w}(\gamma M - wr - w\gamma_1 M^2) > 0, \quad (57)$$

and

$$\frac{1}{1 - w} \left[3(1 - w - \gamma_1^2)M^2 + 3\gamma\gamma_1 M + t(1 - w) - \frac{\gamma^2}{2} - \gamma_1 r \right] \geq 0. \quad (58)$$

For $M \neq (\gamma/2\gamma_1)$ we can express the function $r(M)$ defined by Eq. (54), substitute the obtained expression for $r(M)$ in the existence and stability conditions (54)-(57) and do the analysis in the same way as for $w = 0$. The calculations show that in the domain $r > 0$, FS is stable for $w < 0.5$, when $w = 0.5$ there is a marginal stability, and for $w > 0.5$ the FS-phase is unstable ($0 < w < 1$).

The results can be used to outline the phase diagram and calculate the thermodynamic quantities. This is performed in the way explained in the preceding Sections.

7. Conclusion

We have done an investigation of the M-trigger effect in unconventional ferromagnetic superconductors. This effect due to the $M\psi_1\psi_2$ -coupling term in the GL free energy consists of bringing into existence of superconductivity in a domain of the phase diagram of the system that is entirely in the region of existence of the ferromagnetic

phase. This form of coexistence of unconventional superconductivity and ferromagnetic order is possible for temperatures above and below the critical temperature T_s , which corresponds to the standard phase transition of second order from normal to Meissner phase – usual uniform superconductivity in a zero external magnetic field, which appears outside the domain of existence of ferromagnetic order. Our investigation has been mainly intended to clarify the thermodynamic behaviour at temperatures $T_s < T < T_f$, where the superconductivity cannot appear without the mechanism of M-triggering. We have described the possible ordered phases (FM and FS) in this most interesting temperature interval.

The Cooper pair and crystal anisotropies have also been investigated and their main effects on the thermodynamics of the triggered phase of coexistence have been established. In discussions of concrete real material one should take into account the respective crystal symmetry but the variation of the essential thermodynamic properties with the change of the type of this symmetry is not substantial when the low symmetry and low order (in both M and ψ) γ -term is present in the free energy.

Below the superconducting critical temperature T_s a variety of pure superconducting and mixed phases of coexistence of superconductivity and ferromagnetism exists and the thermodynamic behavior at these relatively low temperatures is more complex than in known cases of improper ferroelectrics. The case $T_f < T_s$ also needs a special investigation.

Our results are referred to the possible uniform superconducting and ferromagnetic states. Vortex and other nonuniform phases need a separate study.

The relation of the present investigation to properties of real ferromagnetic compounds, such as UGe_2 , URhGe , and ZrZn_2 , has been discussed throughout the text. In these real compounds the ferromagnetic critical temperature is much larger than the superconducting critical temperature ($T_f \gg T_s$) and that is why the M-triggering of the spin-triplet superconductivity is very strong. Moreover, the γ_1 -term is important to stabilize the FM order up to the absolute zero (0 K), as is in the known spin-triplet ferromagnetic superconductors. The neglecting [43] of the symmetry conserving γ_1 -term prevents the description of the known real substances of this type. More experimental information about the values of the material parameters (a_s, a_f, \dots) included in the free energy (12) is required in order to outline the thermodynamic behavior and the phase diagram in terms of thermodynamic parameters T and P . In particular, a reliable knowledge about the dependence of the parameters a_s and a_f on the pressure P , the value of the characteristic temperature T_s and the ratio a_s/a_f at zero temperature are of primary interest.

Acknowledgments:

DIU thanks the hospitality of MPI-PKS-Dresden. Financial support by SCENET (Parma) and JINR (Dubna) is also acknowledged.

References

- [1] L. P. Pitaevskii, *Zh. Eksp. Teor. Fiz.* **37** (1959) 1794 [*Sov. Phys. JETP*, **10** (1960) 1267].
- [2] A. J. Leggett, *Rev. Mod. Phys.* **47** (1975) 331.
- [3] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Taylor & Francis, London, 1990).
- [4] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, 2003).
- [5] G. R. Stewart, *Rev. Mod. Phys.* **56** (1984) 755.
- [6] M. Sigrist and K. Ueda, *Rev. Mod. Phys.* **63** (1991) 239.
- [7] V. P. Mineev, K. V. Samokhin, *Introduction to Unconventional Superconductivity* (Gordon and Breach, Amsterdam, 1999).
- [8] M. Sigrist and T. M. Ruce, *Z. Phys. B. - Condensed Matter* **68** (1987) 9.
- [9] J. F. Annett, M. Randeria, and S. R. Renn, *Phys. Rev. B* **38** (1988) 4660.
- [10] G. E. Volovik, *JETP Lett.* **48** (1988) 41 [*Pis'ma Zh. Eksp. Teor. Fiz.* **48** (1988) 39].
- [11] E. J. Blagoeva, G. Busiello, L. De Cesare, Y. T. Millev, I. Rabuffo, and D. I. Uzunov, *Phys. Rev. B* **42** (1990) 6124.
- [12] D. I. Uzunov, in: *Advances in Theoretical Physics*, ed. by E. Caianiello (World Scientific, Singapore, 1990), p. 96.
- [13] D. I. Uzunov, *Theory of Critical Phenomena* (World Scientific, Singapore, 1993).
- [14] J. F. Annett, *Contemp. Physics* **36** (1995) 323.
- [15] D. J. Van Harlingen, *Rev. Mod. Phys.* **67** (1995) 515.
- [16] C. C. Tsuei and J. R. Kirtly, *Rev. Mod. Phys.* **72** (2000) 969.
- [17] G. E. Volovik and L. P. Gor'kov, *JETP Lett.* **39** (1984) 674 [*Pis'ma Zh. Eksp. Teor. Fiz.* **39** (1984) 550].
- [18] G. E. Volovik and L. P. Gor'kov, *Sov. Phys. JETP* **61** (1985) 843 [*Zh. Eksp. Teor. Fiz.* **88** (1985) 1412].
- [19] K. Ueda and T. M. Rice, *Phys. Rev. B* **31** (1985) 7114.
- [20] E. I. Blount, *Phys. Rev. B* **32** (1985) 2935.

- [21] M. Ozaki, K. Machida, and T. Ohmi, *Progr. Theor. Phys.* **74** (1985) 221.
- [22] M. Ozaki, K. Machida, and T. Ohmi, *Progr. Theor. Phys.* **75** (1986) 442.
- [23] S. A. Antonenko and A. I. Sokolov, *Phys. Rev.* B94 (1994) 15901.
- [24] G. Busiello, L. De Cesare, Y. T. Millev, I. Rabuffo, and D. I. Uzunov, *Phys. Rev.* **B43** (1991) 1150.
- [25] G. Busiello, and D. I. Uzunov, *Phys. Rev.* **B42** (1990) 1018.
- [26] S. S. Saxena, P. Agarwal, K. Ahilan, F. M. Grosche, R. K. W. Haselwimmer, M.J. Steiner, E. Pugh, I. R. Walker, S.R. Julian, P. Monthoux, G. G. Lonzarich, A. Huxley, I. Sheikin, D. Braithwaite, and J. Flouquet, *Nature* **406** (2000) 587.
- [27] A. Huxley, I. Sheikin, E. Ressouche, N. Kernavanois, D. Braithwaite, R. Calemczuk, and J. Flouquet, *Phys. Rev.* **B63** (2001) 144519-1.
- [28] N. Tateiwa, T. C. Kobayashi, K. Hanazono, A. Amaya, Y. Haga, R. Settai, and Y. Onuki, *J. Phys. Condensed Matter*, **13** (2001) L17.
- [29] P. Coleman, *Nature* **406** (2000) 580.
- [30] C. Pfleiderer, M. Uhlatz, S. M. Hayden, R. Vollmer, H. v. Löhneysen, N. R. Berhoeft, and G. G. Lonzarich, *Nature* **412** (2001) 58.
- [31] D. Aoki, A. Huxley, E. Ressouche, D. Braithwaite, J. Flouquet, J-P. Brison, E. Lhotel, and C. Paulsen, *Nature* **413** (2001) 613.
- [32] S. V. Vonsovsky, Yu. A. Izyumov, and E. Z. Kurmaev, *Superconductivity of Transition Metals* (Springer Verlag, Berlin, 1982).
- [33] M. B. Maple and F. Fisher (eds), *Superconductivity in Ternary Compounds*, Parts I and II, (Springer Verlag, Berlin, 1982).
- [34] S. K. Sinha, in: *Superconductivity in Magnetic and Exotic Materials*, ed. by T. Matsubara and A. Kotani (Springer Verlag, Berlin, 1984).
- [35] A. Kotani, in: *Superconductivity in Magnetic and Exotic Materials*, ed. by T. Matsubara and A. Kotani (Springer Verlag, Berlin, 1984).
- [36] S. S. Saxena and P. B. Littlewood, *Nature* **412** (2001) 290.
- [37] K. Shimizu, T. Kikura, S. Furomoto, K. Takeda, K. Kontani, Y. Onuki, K. Amaya, *Nature* **412** (2001) 316.
- [38] A. A. Abrikosov and L. P. Gor'kov, *Zh. Eksp. Teor. Fiz.* **39** (1960) 1781 [*Sov. Phys. JETP* **12** (1961) 1243].

- [39] V. L. Ginzburg, *Zh. Eksp. Teor. Fiz.* **31** (1956) 202 [*Sov. Phys. JETP* **4** (1957) 153].
- [40] A. I. Buzdin, L. N. Bulaevskii, S. S. Krotov, *Zh. Eksp. Teor. Fiz.* **85** (1983) 678 [*Sov. Phys. JETP* **58** (1983) 395].
- [41] K. Machida and H. Nakanishi, *Phys. Rev.* **B 30** (1984) 122.
- [42] K. Machida and T. Ohmi, *Phys. Rev. Lett.* **86** (2001) 850.
- [43] M. B. Walker and K. V. Samokhin, *Phys. Rev. Lett.* **88** (2002) 207001-1.
- [44] D. V. Shopova, and D. I. Uzunov, *Phys. Lett. A* **313** (2003) 139.
- [45] D. V. Shopova, and D. I. Uzunov, *J. Phys. Studies* **7** (2003) No 4 (in press); see also, *cond-mat/0305602*
- [46] D. V. Shopova, and D. I. Uzunov, *Compt. Rend. Acad. Bulg. Sciences*, **56** (2003) 35; see also, a corrected version in: *cond-mat/0310016*.
- [47] Yu. M. Gufan and V. I. Torgashev, *Sov. Phys. Solid State* **22** (1980) 951 [*Fiz. Tv. Tela* **22** (1980) 1629].
- [48] Yu. M. Gufan and V. I. Torgashev, *Sov. Phys. Solid State* **23** (1981) 1129.
- [49] L. T. Latush, V. I. Torgashev, and F. Smutny, *Ferroelectrics Letts.* **4** (1985) 37.
- [50] J-C. Tolédano and P. Tolédano, *The Landau Theory of Phase Transitions* (World Scientific, Singapore, 1987).
- [51] Yu. M. Gufan, *Structural Phase Transitions* (Nauka, Moscow, 1982); in Russian.
- [52] R. A. Cowley, *Adv. Phys.* **29** (1980) 1.
- [53] Q. Gu, *Phys. Rev.* **A68** (2003) 025601.
- [54] E. I. Blount, and C. M. Varma, *Phys. Rev. Lett.* **42** (1979) 1079.
- [55] H. S. Greenside, E. I. Blount, and C. M. Varma, *Phys. Rev. Lett.* **46** (1980) 49.
- [56] T. K. Ng, and C. M. Varma, *Phys. Rev. Lett.* **78** (1997) 330.
- [57] C. G. Kuper, M. Revzen, and A. Ron, *Phys. Rev. Lett.* **44** (1980) 1545.
- [58] A. A. Abrikosov, *Zh. Eksp. Teor. Fiz.* **32** (1957) 1442 [*Sov. Phys. JETP*, **5** (1957) 1174].
- [59] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics, II Part* (Pergamon Press, London, 1980) [*Landau-Lifshitz Course in Theoretical Physics, Vol. IX*].
- [60] K. S. Liu, and M. E. Fisher, *J. Low Temp. Phys.* **10** (1973) 655.
- [61] Y. Imry, *J. Phys. C: Cond. Matter Phys.* **8** (1975) 567.